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Brendan Goldsmith
Dublin Institute of Technology, brendan.goldsmith@dit.ie

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A Note on Elongations of Abelian Groups

By

B. Goldsmith

College of Technology

Kevin Street

Dublin

and

Dublin Institute for Advanced Studies

Dublin 4, Ireland.

Throughout this note the word group will refer to an additively written abelian group. For notation and any unexplained terminology we refer to Fuchs [2].

If G is a p-group then for each positive integer n, p^nG is a subgroup of G. We set $p^\omega G = \bigcap p^nG$; $p^\omega G$ is the subgroup of elements of infinite p-height. A reduced p-group G is said to be an ω-elongation of A by B if there exist reduced p-groups A and B such that

(i) $p^\omega G \cong B \neq 0$

(ii) $G/p^\omega G \cong A$.

We call a pair of p-groups (A, B) uniquely ω-elongating if

(i) there exists a p-group G such that $p^\omega G \cong B$

(ii) $B \neq 0$

(iii) $G/p^\omega G \cong A$

(iv) any two such groups are isomorphic.

It follows from the work of Crawley [1] and Hill and Megibben [4] that if A is a direct sum of cyclic p-groups then (A, B) is uniquely ω-elongating for any B for which ω-elongations exist.

The converse problem, viz., if (A, B) is a uniquely ω-elongating pair of p-groups, is A a direct sum of cyclic groups? , is a long standing problem. We consider here a particular case of this problem.

§1. Definitions and preliminary results.

DEFINITION: For an abelian p-group G we define the injective corank of G to be the rank of D/G where D is a minimal divisible group containing G. (Note this rank is independent of the choice of D by a result of Kulikov [6].)

DEFINITION: If H is a subgroup of the p-group G then H is said to be $p^\omega G$-high (or simply high) if H is maximal subject to being a subgroup disjoint from $p^\omega G$.

DEFINITION: If G is a p-group such that all $p^\omega G$-high subgroups are direct sums of cyclic groups then G is said to be a Σ-group.

Proposition 1.1. (Irwin, Peercy and Walker [5]): If one high subgroup of G is a direct sum of cyclic groups, then all high subgroups are isomorphic and G is a Σ-group.

Proposition 1.2. (Megibben [7]): If D is a minimal divisible group containing K and H is a group without elements of infinite height which has a pure subgroup X such that $H/X \cong D/K$, then there exists a group G such that

(i) $G' = K$

(ii) $G/G' \cong H$

(iii) X is a high subgroup of G, where G' is the subgroup of elements of infinite height in G.

The next result gives necessary and sufficient conditions for ω-elongations to exist.

Theorem 1.3. (Pierce [8]): Let A and B be reduced abelian p-groups such that $p^{\omega}A = 0$. Then there is a group G such that $p^{\omega}G \cong B$, $G/p^{\omega}G \cong A$ if and only if

$$\dim (p^n A[p]) \geq \dim (B/pB) \qquad \text{for all } n < \omega .$$

§2. Elongations of p-groups.

Theorem 2.1: Let A and B be reduced p-groups such that ω-elongations of A by B exist. Then, if the final rank of A equals the injective corank of B and A is not a direct sum of cyclic groups, the pair (A, B) is not uniquely ω-elongating.

Proof:

Since A is not a direct sum of cyclic groups it contains a pure subgroup X such that X is pure, dense, the rank of A/X equals the final rank of A and X is not a direct sum of cyclic groups. (If not by a result of Hill [3], A would itself be a direct sum of cyclic groups; cf. Warfield [9].)

If C is a lower basic subgroup of A then rk. (A/C) = final rk. A = rk. (D/B) where D is a minimal divisible group containing B. Then by Prop. 1.2. there exists a p-group G with $p^{\omega}G \cong B$, $G/p^{\omega}G \cong A$ and C is high in G. Since C is a direct sum of cyclic groups, it follows from Prop. 1.1. that G is a Σ-group.

However rk. (A/X) = final rk.A = rk. (D/B) and so by Prop. 1.2. there exists a p-group H with $p^{\omega}H \cong B$, $H/p^{\omega}H \cong A$ and X is high in H. Since X is not a direct sum of cyclic groups it is clear from Prop. 1.1. that H is not a Σ-group. Thus $H \ncong G$, i.e. the pair (A, B) is not uniquely ω-elongating.

<u>Corollary 2.2.</u>: If A is not a direct sum of cyclic groups and ω-elongations of A by B exist then there is a cardinal m such that ω-elongations of A by $\bigoplus_m B$ exist but these are not unique.

Proof:

By Th. 1.3. Dim $B/pB \leq$ final rk. A. Then if m equals the final rank of A we have Dim $\left(\bigoplus_m B/p(\bigoplus_m B)\right)$ = final rk. A. So by Th. 1.3. ω-elongations of A by $\bigoplus_m B$ exist. Moreover injective corank $\bigoplus_m B$ = final rk. A and so the result follows from 2.1.

We remark in conclusion that the above may be easily generalised to elongations of modules over a complete discrete valuation ring.

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College of Technology,
Kevin Street, Dublin.

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Advanced Studies.